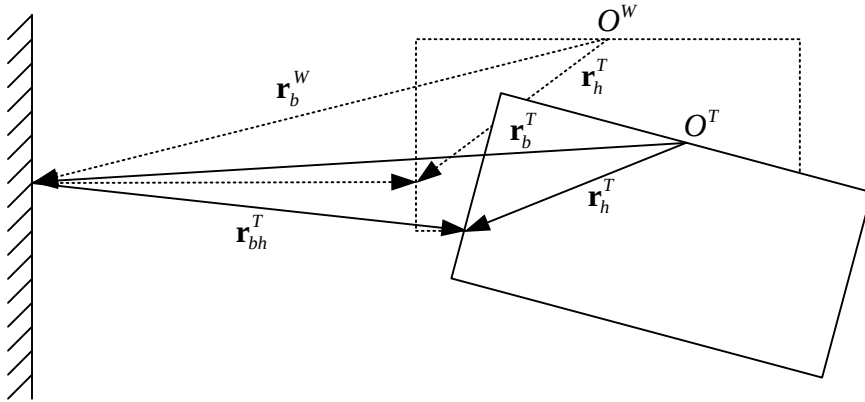


Analytic Jacobian

Brad Thoen, Staff Engineer
MTS Systems Corporation
28-Dec-05

See *Coordinate Transforms* for background and definition of terms.

Inline Actuator Case



For each actuator i :

$$\mathbf{r}_{bhi}^T = \mathbf{r}_{hi}^T - \mathbf{r}_{bi}^T = \mathbf{r}_{hi}^T - \mathbf{T}_{W \rightarrow T} \mathbf{r}_{bi}^W$$

$$\mathbf{T}_{T \rightarrow W} = \begin{bmatrix} \mathbf{R}_{T \rightarrow W} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{where } \mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{T}_{W \rightarrow T} = \mathbf{T}_{T \rightarrow W}^{-1} = \begin{bmatrix} \mathbf{R}_{T \rightarrow W}^{-1} & -\mathbf{R}_{T \rightarrow W}^{-1} \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}_{T \rightarrow W}^{-1} = \mathbf{R}'_{T \rightarrow W}$$

$$\mathbf{r}_{bhi}^T = \mathbf{r}_{hi}^T - \mathbf{R}_{W \rightarrow T} (\mathbf{r}_{bi}^W - \mathbf{p})$$

$$\ell_i = \|\mathbf{r}_{bhi}^T\| = \sqrt{\mathbf{r}_{bhi}^T|_x^2 + \mathbf{r}_{bhi}^T|_y^2 + \mathbf{r}_{bhi}^T|_z^2}$$

Element i,j of jacobian $\mathbf{J}$ corresponding to actuator i and DOF j :

$$\mathbf{J}_{ij} = \frac{\partial \ell_i}{\partial q_j} = \frac{1}{2} \cdot \frac{1}{\sqrt{\mathbf{r}_{bhi}^T|_x^2 + \mathbf{r}_{bhi}^T|_y^2 + \mathbf{r}_{bhi}^T|_z^2}} \cdot 2 \left[\mathbf{r}_{bhi}^T|_x \frac{\partial \mathbf{r}_{bhi}^T|_x}{\partial q_j} + \mathbf{r}_{bhi}^T|_y \frac{\partial \mathbf{r}_{bhi}^T|_y}{\partial q_j} + \mathbf{r}_{bhi}^T|_z \frac{\partial \mathbf{r}_{bhi}^T|_z}{\partial q_j} \right]$$

$$\mathbf{J}_{ij} = \frac{1}{\|\mathbf{r}_{bh_i}^T\|} \cdot \begin{bmatrix} \mathbf{r}_{bh_i}^T \big|_x & \mathbf{r}_{bh_i}^T \big|_y & \mathbf{r}_{bh_i}^T \big|_z \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial \mathbf{r}_{bh_i}^T \big|_x}{\partial q_j} \\ \frac{\partial \mathbf{r}_{bh_i}^T \big|_y}{\partial q_j} \\ \frac{\partial \mathbf{r}_{bh_i}^T \big|_z}{\partial q_j} \end{bmatrix} = \frac{\mathbf{r}_{bh_i}^T}{\|\mathbf{r}_{bh_i}^T\|} \cdot \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$$

$$\boxed{\mathbf{J}_{ij} = e_{bh_i}^T \cdot \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}}$$

Derivatives $\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$ for the general case:

$$\boxed{\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \frac{\partial}{\partial q_j} [\mathbf{r}_{h_i}^T - \mathbf{R}_{W \rightarrow T}(\mathbf{r}_{b_i}^W - \mathbf{p})]}$$

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j}(\mathbf{r}_{b_i}^W - \mathbf{p}) - \mathbf{R}_{W \rightarrow T} \left(\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} - \frac{\partial \mathbf{p}}{\partial q_j} \right)$$

In inline actuator case, $\mathbf{r}_{h_i}^T$ and $\mathbf{r}_{b_i}^W$ are constants, so terms $\frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j}$ and $\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j}$ are zero, yielding derivatives

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = 0 - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j}(\mathbf{r}_{b_i}^W - \mathbf{p}) - \mathbf{R}_{W \rightarrow T} \left(0 - \frac{\partial \mathbf{p}}{\partial q_j} \right) = \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial q_j} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j}(\mathbf{r}_{b_i}^W - \mathbf{p})$$

Evaluating $\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$ for $q_j = x, y, z, \theta_x, \theta_y, \theta_z$:

$$\begin{aligned} \frac{\partial \mathbf{r}_{bh_i}^T}{\partial x} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial x} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial x}(\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 0 \cdot (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \\ \frac{\partial \mathbf{r}_{bh_i}^T}{\partial y} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial y} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial y}(\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 0 \cdot (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ \frac{\partial \mathbf{r}_{bh_i}^T}{\partial z} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial z} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial z}(\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - 0 \cdot (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{r}_{bh_i}^T}{\partial \theta_x} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial \theta_x} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \cdot 0 - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} (\mathbf{r}_{b_i}^W - \mathbf{p}) = -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} (\mathbf{r}_{b_i}^W - \mathbf{p}) \\ \frac{\partial \mathbf{r}_{bh_i}^T}{\partial \theta_y} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial \theta_y} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \cdot 0 - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} (\mathbf{r}_{b_i}^W - \mathbf{p}) = -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} (\mathbf{r}_{b_i}^W - \mathbf{p}) \\ \frac{\partial \mathbf{r}_{bh_i}^T}{\partial \theta_z} &= \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{p}}{\partial \theta_z} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} (\mathbf{r}_{b_i}^W - \mathbf{p}) = \mathbf{R}_{W \rightarrow T} \cdot 0 - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} (\mathbf{r}_{b_i}^W - \mathbf{p}) = -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} (\mathbf{r}_{b_i}^W - \mathbf{p})\end{aligned}$$

Note in the above that

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} = 0 \quad \text{for } q_j = x, y, z$$

and

$$\frac{\partial \mathbf{p}}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \frac{\partial \mathbf{p}}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial \theta_x} = \frac{\partial \mathbf{p}}{\partial \theta_y} = \frac{\partial \mathbf{p}}{\partial \theta_z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rotation matrix $\mathbf{R}_{W \rightarrow T}$ (also known as the Direction Cosine Matrix (DCM)) and its derivatives, corresponding to the SAE standard for Euler angle rotation (first yaw, then pitch, then roll), are derived in the paper titled *Euler Rotations* and are shown below:

$$\mathbf{R}_{W \rightarrow T} = \begin{bmatrix} cy \cdot cz & cy \cdot sz & -sy \\ sx \cdot sy \cdot cz - cx \cdot sz & sx \cdot sy \cdot sz + cx \cdot cz & sx \cdot cy \\ cx \cdot sy \cdot cz + sx \cdot sz & cx \cdot sy \cdot sz - sx \cdot cz & cx \cdot cy \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} = \begin{bmatrix} 0 & 0 & 0 \\ cx \cdot sy \cdot cz + sx \cdot sz & cx \cdot sy \cdot sz - sx \cdot cz & cx \cdot cy \\ -sx \cdot sy \cdot cz + cx \cdot sz & -sx \cdot sy \cdot sz - cx \cdot cz & -sx \cdot cy \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} = \begin{bmatrix} -sy \cdot cz & -sy \cdot sz & -cy \\ sx \cdot cy \cdot cz & sx \cdot cy \cdot sz & -sx \cdot sy \\ cx \cdot cy \cdot cz & cx \cdot cy \cdot sz & -cx \cdot sy \end{bmatrix}$$

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} = \begin{bmatrix} -cy \cdot sz & cy \cdot cz & 0 \\ -sx \cdot sy \cdot sz - cx \cdot cz & sx \cdot sy \cdot cz - cx \cdot sz & 0 \\ -cx \cdot sy \cdot sz + sx \cdot cz & cx \cdot sy \cdot cz + sx \cdot sz & 0 \end{bmatrix}$$

where

$$\begin{aligned}cx &= \cos \theta_x & sx &= \sin \theta_x \\ cy &= \cos \theta_y & sy &= \sin \theta_y \\ cz &= \cos \theta_z & sz &= \sin \theta_z\end{aligned}$$

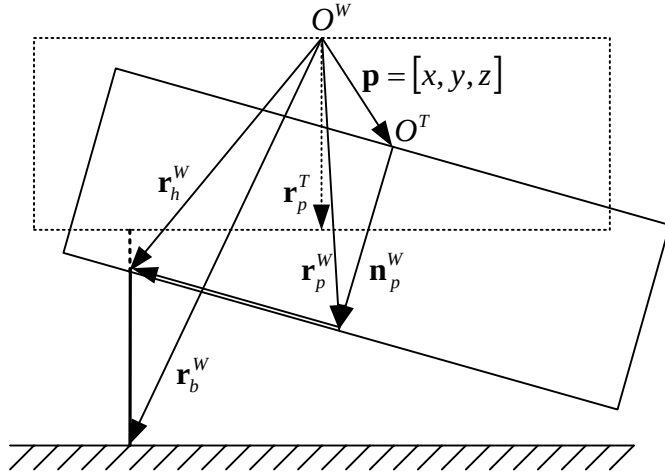
Summary for inline actuator case:

Element i,j of jacobian $\mathbf{J}$ corresponding to actuator i and DOF j is given by:

$$\mathbf{J}_{ij} = e_{bh_i}^T \cdot \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$$

where

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \begin{cases} \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & (q_j = x) \\ \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & (q_j = y) \\ \mathbf{R}_{W \rightarrow T} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & (q_j = z) \\ -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_x} (\mathbf{r}_{b_i}^W - \mathbf{p}) & (q_j = \theta_x) \\ -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_y} (\mathbf{r}_{b_i}^W - \mathbf{p}) & (q_j = \theta_y) \\ -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial \theta_z} (\mathbf{r}_{b_i}^W - \mathbf{p}) & (q_j = \theta_z) \end{cases}$$

Sliding Head Actuator Case

Unlike the inline actuator case where table pivot location vectors $\mathbf{r}_{h_i}^T$ are constant, in the sliding head actuator case they vary with translation and rotation and are determined by finding the intersection of the actuator head with the plane of the table:

$$\mathbf{r}_{p_i}^W = \mathbf{R}_{T \rightarrow W} \mathbf{r}_{p_i}^T + \mathbf{p} \quad (\text{radius vector from table origin to sliding surface})$$

$$\mathbf{n}_{p_i}^W = \mathbf{r}_{p_i}^W - \mathbf{p} = \mathbf{R}_{T \rightarrow W} \mathbf{r}_{p_i}^T \quad (\text{normal to the sliding plane})$$

Radius vectors $\mathbf{r}_{p_i}^T$ from the origin to the sliding surface are given by:

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} \mathbf{r}_{h_i}^T|_{x0} \\ 0 \\ 0 \end{bmatrix} \quad \text{for x-facing actuators,}$$

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} 0 \\ \mathbf{r}_{h_i}^T|_{y0} \\ 0 \end{bmatrix} \quad \text{for y-facing actuators}$$

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} 0 \\ 0 \\ \mathbf{r}_{h_i}^T|_{z0} \end{bmatrix} \quad \text{for z-facing actuators}$$

Note that the zero subscript in the above expressions denotes the nominal $\mathbf{r}_{h_i}^T$ before translation and rotation.

The plane equation is given by:

$$0 = \mathbf{n}_{p_i}^W|_x \left(\mathbf{r}_{h_i}^W|_x - \mathbf{r}_{p_i}^W|_x \right) + \mathbf{n}_{p_i}^W|_y \left(\mathbf{r}_{h_i}^W|_y - \mathbf{r}_{p_i}^W|_y \right) + \mathbf{n}_{p_i}^W|_z \left(\mathbf{r}_{h_i}^W|_z - \mathbf{r}_{p_i}^W|_z \right)$$

For x-facing actuators, solve for x component of $\mathbf{r}_{h_i}^W$ (the y and z components equal that of $\mathbf{r}_{b_i}^W$):

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{p_i}^W \big|_x - \left\{ \mathbf{n}_{p_i}^W \big|_y \left(\mathbf{r}_{h_i}^W \big|_y - \mathbf{r}_{p_i}^W \big|_y \right) + \mathbf{n}_{p_i}^W \big|_z \left(\mathbf{r}_{h_i}^W \big|_z - \mathbf{r}_{p_i}^W \big|_z \right) \right\} / \mathbf{n}_{p_i}^W \big|_x \\ \mathbf{r}_{b_i}^W \big|_y \\ \mathbf{r}_{b_i}^W \big|_z \end{bmatrix}$$

For y-facing actuators, solve for y component of $\mathbf{r}_{h_i}^W$ (the x and z components equal that of $\mathbf{r}_{b_i}^W$):

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W \big|_x \\ \mathbf{r}_{p_i}^W \big|_y - \left\{ \mathbf{n}_{p_i}^W \big|_x \left(\mathbf{r}_{h_i}^W \big|_x - \mathbf{r}_{p_i}^W \big|_x \right) + \mathbf{n}_{p_i}^W \big|_z \left(\mathbf{r}_{h_i}^W \big|_z - \mathbf{r}_{p_i}^W \big|_z \right) \right\} / \mathbf{n}_{p_i}^W \big|_y \\ \mathbf{r}_{b_i}^W \big|_z \end{bmatrix}$$

For z-facing actuators, solve for z component of $\mathbf{r}_{h_i}^W$ (the x and y components equal that of $\mathbf{r}_{b_i}^W$):

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W \big|_x \\ \mathbf{r}_{b_i}^W \big|_y \\ \mathbf{r}_{p_i}^W \big|_z - \left\{ \mathbf{n}_{p_i}^W \big|_x \left(\mathbf{r}_{h_i}^W \big|_x - \mathbf{r}_{p_i}^W \big|_x \right) + \mathbf{n}_{p_i}^W \big|_y \left(\mathbf{r}_{h_i}^W \big|_y - \mathbf{r}_{p_i}^W \big|_y \right) \right\} / \mathbf{n}_{p_i}^W \big|_z \end{bmatrix}$$

Now that $\mathbf{r}_{h_i}^W$ is known, find the derivatives $\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$ using the general equation

$$\begin{aligned} \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} &= \frac{\partial}{\partial q_j} [\mathbf{r}_{h_i}^T - \mathbf{R}_{W \rightarrow T} (\mathbf{r}_{b_i}^W - \mathbf{p})] = \frac{\partial}{\partial q_j} [\mathbf{R}_{W \rightarrow T} (\mathbf{r}_{h_i}^W - \mathbf{p}) - \mathbf{R}_{W \rightarrow T} (\mathbf{r}_{b_i}^W - \mathbf{p})] \\ &\Rightarrow \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \frac{\partial}{\partial q_j} [\mathbf{R}_{W \rightarrow T} (\mathbf{r}_{h_i}^W - \mathbf{r}_{b_i}^W)] = \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} (\mathbf{r}_{h_i}^W - \mathbf{r}_{b_i}^W) + \mathbf{R}_{W \rightarrow T} \left(\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} - \frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} \right) \end{aligned}$$

Note that in sliding head actuator case, $\mathbf{r}_{b_i}^W$ is constant, so term $\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j}$ is zero:

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} (\mathbf{r}_{h_i}^W - \mathbf{r}_{b_i}^W) + \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j}$$

In the above equation, only $\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j}$ is unknown. To find it, differentiate the plane equation:

$$\begin{aligned}
0 &= \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \Big|_x \left(\mathbf{r}_{h_i}^w \Big|_x - \mathbf{r}_{p_i}^w \Big|_x \right) + \mathbf{n}_{p_i}^w \Big|_x \left(\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x - \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} \Big|_x \right) \\
&+ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \Big|_y \left(\mathbf{r}_{h_i}^w \Big|_y - \mathbf{r}_{p_i}^w \Big|_y \right) + \mathbf{n}_{p_i}^w \Big|_y \left(\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y - \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} \Big|_y \right) \\
&+ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \Big|_z \left(\mathbf{r}_{h_i}^w \Big|_z - \mathbf{r}_{p_i}^w \Big|_z \right) + \mathbf{n}_{p_i}^w \Big|_z \left(\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z - \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} \Big|_z \right) \\
\Rightarrow 0 &= \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w \right) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_x \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x + \mathbf{n}_{p_i}^w \Big|_y \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y + \mathbf{n}_{p_i}^w \Big|_z \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z
\end{aligned}$$

Solve for $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x$, $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y$, and $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z$:

$$\begin{aligned}
\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x &= - \left\{ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w \right) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_y \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y + \mathbf{n}_{p_i}^w \Big|_z \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z \right\} / \mathbf{n}_{p_i}^w \Big|_x \\
\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y &= - \left\{ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w \right) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_x \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x + \mathbf{n}_{p_i}^w \Big|_z \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z \right\} / \mathbf{n}_{p_i}^w \Big|_y \\
\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z &= - \left\{ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w \right) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_x \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x + \mathbf{n}_{p_i}^w \Big|_y \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y \right\} / \mathbf{n}_{p_i}^w \Big|_z
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} &= \frac{\partial (\mathbf{R}_{T \rightarrow W} \mathbf{r}_{p_i}^T)}{\partial q_j} = \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{p_i}^T + \mathbf{R}_{T \rightarrow W} \frac{\partial \mathbf{r}_{p_i}^T}{\partial q_j} = \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{p_i}^T \quad \text{because} \quad \frac{\partial \mathbf{r}_{p_i}^T}{\partial q_j} = 0. \\
\Rightarrow \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} &= \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{p_i}^T = \begin{cases} \vec{\mathbf{0}} & (q_j = x) \\ \vec{\mathbf{0}} & (q_j = y) \\ \vec{\mathbf{0}} & (q_j = z) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_x} \mathbf{r}_{p_i}^T & (q_j = \theta_x) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_y} \mathbf{r}_{p_i}^T & (q_j = \theta_y) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_z} \mathbf{r}_{p_i}^T & (q_j = \theta_z) \end{cases}
\end{aligned}$$

and

$$\frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} = \frac{\partial (\mathbf{n}_{p_i}^W + \mathbf{p})}{\partial q_j} = \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} + \frac{\partial \mathbf{p}}{\partial q_j} = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & (q_j = x) \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & (q_j = y) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & (q_j = z) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_x} \mathbf{r}_{p_i}^T & (q_j = \theta_x) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_y} \mathbf{r}_{p_i}^T & (q_j = \theta_y) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_z} \mathbf{r}_{p_i}^T & (q_j = \theta_z) \end{cases}$$

Note in the above that

$$\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} = 0 \quad \text{for } q_j = x, y, z$$

$$\frac{\partial \mathbf{p}}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \frac{\partial \mathbf{p}}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial \theta_x} = \frac{\partial \mathbf{p}}{\partial \theta_y} = \frac{\partial \mathbf{p}}{\partial \theta_z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Derivatives $\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j}$ for x-facing actuators are:

$$\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{h_i}^W \Big|_x \\ \mathbf{r}_{b_i}^W \Big|_y \\ \mathbf{r}_{b_i}^W \Big|_z \end{bmatrix} = \begin{bmatrix} - \left\{ \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} \bullet (\mathbf{r}_{h_i}^W - \mathbf{r}_{p_i}^W) - \mathbf{n}_{p_i}^W \bullet \frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} + \mathbf{n}_{p_i}^W \Big|_y \frac{\partial \mathbf{r}_{h_i}^W \Big|_y}{\partial q_j} + \mathbf{n}_{p_i}^W \Big|_z \frac{\partial \mathbf{r}_{h_i}^W \Big|_z}{\partial q_j} \right\} / \mathbf{n}_{p_i}^W \Big|_x \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \begin{bmatrix} \left\{ \mathbf{n}_{p_i}^W \bullet \frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} \bullet (\mathbf{r}_{h_i}^W - \mathbf{r}_{p_i}^W) \right\} / \mathbf{n}_{p_i}^W \Big|_x \\ 0 \\ 0 \end{bmatrix} \quad \text{because } \frac{\partial \mathbf{r}_{h_i}^W \Big|_y}{\partial q_j} = 0 \text{ and } \frac{\partial \mathbf{r}_{h_i}^W \Big|_z}{\partial q_j} = 0.$$

Derivatives $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j}$ for y-facing actuators are:

$$\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{b_i}^w \Big|_x \\ \mathbf{r}_{h_i}^w \Big|_y \\ \mathbf{r}_{b_i}^w \Big|_z \end{bmatrix} = \begin{bmatrix} 0 \\ - \left\{ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet (\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_x \frac{\partial \mathbf{r}_{h_i}^w \Big|_x}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_z \frac{\partial \mathbf{r}_{h_i}^w \Big|_z}{\partial q_j} \right\} / \mathbf{n}_{p_i}^w \Big|_y \\ 0 \end{bmatrix}$$

$$\Rightarrow \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} = \begin{bmatrix} 0 \\ \left\{ \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet (\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w) \right\} / \mathbf{n}_{p_i}^w \Big|_y \\ 0 \end{bmatrix} \text{ because } \partial \mathbf{r}_{h_i}^w \Big|_x = 0 \text{ and } \partial \mathbf{r}_{h_i}^w \Big|_z = 0.$$

Derivatives $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j}$ for z-facing actuators are:

$$\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{b_i}^w \Big|_x \\ \mathbf{r}_{b_i}^w \Big|_y \\ \mathbf{r}_{h_i}^w \Big|_z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ - \left\{ \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet (\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w) - \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_x \frac{\partial \mathbf{r}_{h_i}^w \Big|_x}{\partial q_j} + \mathbf{n}_{p_i}^w \Big|_y \frac{\partial \mathbf{r}_{h_i}^w \Big|_y}{\partial q_j} \right\} / \mathbf{n}_{p_i}^w \Big|_z \end{bmatrix}$$

$$\Rightarrow \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} = \begin{bmatrix} 0 \\ 0 \\ \left\{ \mathbf{n}_{p_i}^w \bullet \frac{\partial \mathbf{r}_{p_i}^w}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^w}{\partial q_j} \bullet (\mathbf{r}_{h_i}^w - \mathbf{r}_{p_i}^w) \right\} / \mathbf{n}_{p_i}^w \Big|_z \end{bmatrix} \text{ because } \partial \mathbf{r}_{h_i}^w \Big|_x = 0 \text{ and } \partial \mathbf{r}_{h_i}^w \Big|_y = 0.$$

Summary for sliding head actuator case:

Element i,j of jacobian $\mathbf{J}$ corresponding to actuator i and DOF j is given by:

$$\mathbf{J}_{ij} = \mathbf{e}_{bh_i}^T \cdot \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$$

where

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \begin{cases} \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} & (q_j = x, y, z) \\ \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} (\mathbf{r}_{h_i}^W - \mathbf{r}_{b_i}^W) + \mathbf{R}_{W \rightarrow T} \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} & (q_j = \theta_x, \theta_y, \theta_z) \end{cases}$$

Common quantities for x-, y-, and z-facing actuators:

$$\mathbf{n}_{p_i}^W = \mathbf{R}_{T \rightarrow W} \mathbf{r}_{p_i}^T$$

$$\mathbf{r}_{p_i}^W = \mathbf{n}_{p_i}^W + \mathbf{p}$$

$$\frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} = \begin{cases} \vec{\mathbf{0}} & (q_j = x) \\ \vec{\mathbf{0}} & (q_j = y) \\ \vec{\mathbf{0}} & (q_j = z) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_x} \mathbf{r}_{p_i}^T & (q_j = \theta_x) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_y} \mathbf{r}_{p_i}^T & (q_j = \theta_y) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_z} \mathbf{r}_{p_i}^T & (q_j = \theta_z) \end{cases}$$

$$\frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & (q_j = x) \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & (q_j = y) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & (q_j = z) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_x} \mathbf{r}_{p_i}^T & (q_j = \theta_x) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_y} \mathbf{r}_{p_i}^T & (q_j = \theta_y) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_z} \mathbf{r}_{p_i}^T & (q_j = \theta_z) \end{cases}$$

Summary for sliding head actuator case (continued):

For x-facing actuators:

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} \mathbf{r}_{h_i}^T|_{x0} \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{p_i}^W|_x - \left\{ \mathbf{n}_{p_i}^W|_y \left(\mathbf{r}_{h_i}^W|_y - \mathbf{r}_{p_i}^W|_y \right) + \mathbf{n}_{p_i}^W|_z \left(\mathbf{r}_{h_i}^W|_z - \mathbf{r}_{p_i}^W|_z \right) \right\} / \mathbf{n}_{p_i}^W|_x} \\ \mathbf{r}_{b_i}^W|_y \\ \mathbf{r}_{b_i}^W|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \begin{bmatrix} \left\{ \mathbf{n}_{p_i}^W \bullet \frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^W - \mathbf{r}_{p_i}^W \right) \right\} / \mathbf{n}_{p_i}^W|_x} \\ 0 \\ 0 \end{bmatrix}$$

For y-facing actuators:

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} 0 \\ \mathbf{r}_{h_i}^T|_{y0} \\ 0 \end{bmatrix}$$

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W|_x \\ \mathbf{r}_{p_i}^W|_y - \left\{ \mathbf{n}_{p_i}^W|_x \left(\mathbf{r}_{h_i}^W|_x - \mathbf{r}_{p_i}^W|_x \right) + \mathbf{n}_{p_i}^W|_z \left(\mathbf{r}_{h_i}^W|_z - \mathbf{r}_{p_i}^W|_z \right) \right\} / \mathbf{n}_{p_i}^W|_y} \\ \mathbf{r}_{b_i}^W|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \begin{bmatrix} 0 \\ \left\{ \mathbf{n}_{p_i}^W \bullet \frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^W - \mathbf{r}_{p_i}^W \right) \right\} / \mathbf{n}_{p_i}^W|_y} \\ 0 \end{bmatrix}$$

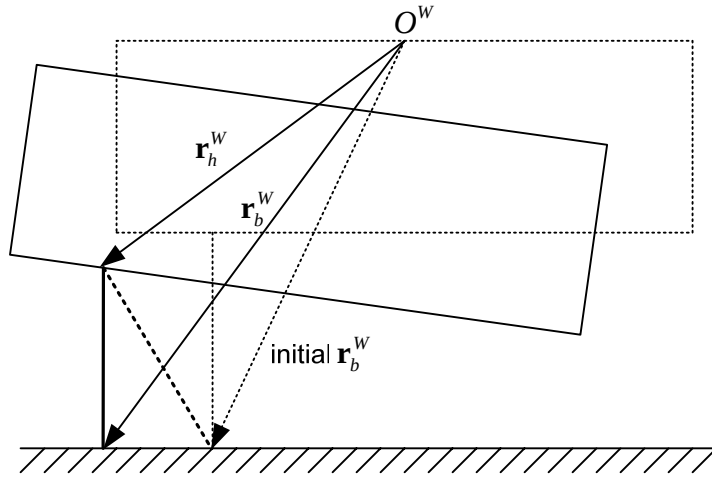
Summary for sliding head actuator case (continued):

For z-facing actuators:

$$\mathbf{r}_{p_i}^T = \begin{bmatrix} 0 \\ 0 \\ \mathbf{r}_{h_i}^T|_{z0} \end{bmatrix}$$

$$\mathbf{r}_{h_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W|_x \\ \mathbf{r}_{b_i}^W|_y \\ \mathbf{r}_{p_i}^W|_z - \left\{ \mathbf{n}_{p_i}^W|_x \left(\mathbf{r}_{h_i}^W|_x - \mathbf{r}_{p_i}^W|_x \right) + \mathbf{n}_{p_i}^W|_y \left(\mathbf{r}_{h_i}^W|_y - \mathbf{r}_{p_i}^W|_y \right) \right\} / \mathbf{n}_{p_i}^W|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \begin{bmatrix} 0 \\ 0 \\ \left\{ \mathbf{n}_{p_i}^W \bullet \frac{\partial \mathbf{r}_{p_i}^W}{\partial q_j} - \frac{\partial \mathbf{n}_{p_i}^W}{\partial q_j} \bullet \left(\mathbf{r}_{h_i}^W - \mathbf{r}_{p_i}^W \right) \right\} / \mathbf{n}_{p_i}^W|_z \end{bmatrix}$$

Sliding Base Actuator Case

Unlike the inline actuator case where base pivot location vectors $\mathbf{r}_{b_i}^T$ are constant, in the sliding base actuator case they vary with translation and rotation and are determined by transforming the table pivot location vectors $\mathbf{r}_{h_i}^T$ to the world coordinate system and inserting the x- and y-components of $\mathbf{r}_{h_i}^W$ into base pivot location vector $\mathbf{r}_{b_i}^W$:

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W | x \\ \mathbf{r}_{h_i}^W | y \\ \mathbf{r}_{h_i}^W | z \end{bmatrix} \quad \text{for x-facing actuators}$$

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{h_i}^W | x \\ \mathbf{r}_{b_i}^W | y \\ \mathbf{r}_{h_i}^W | z \end{bmatrix} \quad \text{for y-facing actuators}$$

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{h_i}^W | x \\ \mathbf{r}_{h_i}^W | y \\ \mathbf{r}_{b_i}^W | z \end{bmatrix} \quad \text{for z-facing actuators}$$

Now that $\mathbf{r}_{b_i}^W$ is known, find the derivatives $\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$ using the general equation

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = \frac{\partial}{\partial q_j} [\mathbf{r}_{h_i}^T - \mathbf{R}_{W \rightarrow T}(\mathbf{r}_{b_i}^W - \mathbf{p})] = \frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j} - \frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j}(\mathbf{r}_{b_i}^W - \mathbf{p}) - \mathbf{R}_{W \rightarrow T} \left(\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} - \frac{\partial \mathbf{p}}{\partial q_j} \right)$$

Note that in sliding head actuator case, $\mathbf{r}_{h_i}^T$ is constant, so term $\frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j}$ is zero:

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = -\frac{\partial \mathbf{R}_{w \rightarrow T}}{\partial q_j} (\mathbf{r}_{b_i}^w - \mathbf{p}) - \mathbf{R}_{w \rightarrow T} \left(\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j} - \frac{\partial \mathbf{p}}{\partial q_j} \right)$$

In the above equation, only $\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j}$ is unknown. Because $\mathbf{r}_{b_i}^w$ has components equal to that of $\mathbf{r}_{h_i}^w$,

it will be necessary to derive $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j}$:

$$\begin{aligned} \mathbf{r}_{h_i}^w &= \mathbf{R}_{T \rightarrow W} \mathbf{r}_{h_i}^T + \mathbf{p} \\ \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} &= \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{h_i}^T + \mathbf{R}_{T \rightarrow W} \frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j} + \frac{\partial \mathbf{p}}{\partial q_j} \\ \Rightarrow \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} &= \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{h_i}^T + \frac{\partial \mathbf{p}}{\partial q_j} \end{aligned}$$

Note that in the above, $\mathbf{r}_{h_i}^T$ is constant, so $\frac{\partial \mathbf{r}_{h_i}^T}{\partial q_j}$ is zero.

Evaluating $\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j}$ for $q_j = x, y, z, \theta_x, \theta_y, \theta_z$:

$$\frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} = \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{h_i}^T + \frac{\partial \mathbf{p}}{\partial q_j} = \begin{cases} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} & (q_j = x) \\ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} & (q_j = y) \\ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} & (q_j = z) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_x} \mathbf{r}_{h_i}^T & (q_j = \theta_x) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_y} \mathbf{r}_{h_i}^T & (q_j = \theta_y) \\ \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial \theta_z} \mathbf{r}_{h_i}^T & (q_j = \theta_z) \end{cases}$$

where it is noted that

$$\frac{\partial \mathbf{R}_{w \rightarrow T}}{\partial q_j} = 0 \quad \text{for } q_j = x, y, z$$

and

$$\frac{\partial \mathbf{p}}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow \frac{\partial \mathbf{p}}{\partial x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \frac{\partial \mathbf{p}}{\partial \theta_x} = \frac{\partial \mathbf{p}}{\partial \theta_y} = \frac{\partial \mathbf{p}}{\partial \theta_z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Derivatives $\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j}$ for x-facing actuators are:

$$\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{b_i}^w \Big|_x \\ \mathbf{r}_{h_i}^w \Big|_y \\ \mathbf{r}_{h_i}^w \Big|_z \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y \\ \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z \end{bmatrix}$$

Derivatives $\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j}$ for y-facing actuators are:

$$\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{h_i}^w \Big|_x \\ \mathbf{r}_{b_i}^w \Big|_y \\ \mathbf{r}_{h_i}^w \Big|_z \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x \\ 0 \\ \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_z \end{bmatrix}$$

Derivatives $\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j}$ for z-facing actuators are:

$$\frac{\partial \mathbf{r}_{b_i}^w}{\partial q_j} = \frac{\partial}{\partial q_j} \begin{bmatrix} \mathbf{r}_{h_i}^w \Big|_x \\ \mathbf{r}_{h_i}^w \Big|_y \\ \mathbf{r}_{b_i}^w \Big|_z \end{bmatrix} = \begin{bmatrix} \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_x \\ \frac{\partial \mathbf{r}_{h_i}^w}{\partial q_j} \Big|_y \\ 0 \end{bmatrix}$$

Summary for sliding base actuator case:

Element i,j of jacobian $\mathbf{J}$ corresponding to actuator i and DOF j is given by:

$$\mathbf{J}_{ij} = \mathbf{e}_{bh_i}^T \cdot \frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j}$$

where

$$\frac{\partial \mathbf{r}_{bh_i}^T}{\partial q_j} = -\frac{\partial \mathbf{R}_{W \rightarrow T}}{\partial q_j} (\mathbf{r}_{b_i}^W - \mathbf{p}) - \mathbf{R}_{W \rightarrow T} \left(\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} - \frac{\partial \mathbf{p}}{\partial q_j} \right)$$

Common for x-, y-, and z-facing actuators:

$$\frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} = \frac{\partial \mathbf{R}_{T \rightarrow W}}{\partial q_j} \mathbf{r}_{h_i}^T + \frac{\partial \mathbf{p}}{\partial q_j}$$

For x-facing actuators:

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{b_i}^W \Big|_x \\ \mathbf{r}_{h_i}^W \Big|_y \\ \mathbf{r}_{h_i}^W \Big|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} = \begin{bmatrix} 0 \\ \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_y \\ \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_z \end{bmatrix}$$

For y-facing actuators:

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{h_i}^W \Big|_x \\ \mathbf{r}_{b_i}^W \Big|_y \\ \mathbf{r}_{h_i}^W \Big|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} = \begin{bmatrix} \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_x \\ 0 \\ \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_z \end{bmatrix}$$

For z-facing actuators:

$$\mathbf{r}_{b_i}^W = \begin{bmatrix} \mathbf{r}_{h_i}^W \Big|_x \\ \mathbf{r}_{h_i}^W \Big|_y \\ \mathbf{r}_{b_i}^W \Big|_z \end{bmatrix}$$

$$\frac{\partial \mathbf{r}_{b_i}^W}{\partial q_j} = \begin{bmatrix} \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_x \\ \frac{\partial \mathbf{r}_{h_i}^W}{\partial q_j} \Big|_y \\ 0 \end{bmatrix}$$